The Shannon machine

(Colloquium *"The Legacy of Claude Shannon"*)

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13 December 2016

[*] Daniel Graça was partially supported by *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI via Instituto de Telecomunicações through the FCT project UID/EEA/50008/2013.

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This talk discusses some less well-known results of Shannon about mathematical models for (analog) computation.

What is a computer?

Antikythera mechanism

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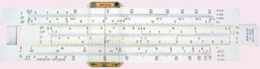
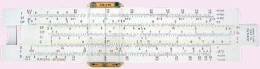


Slide Rule

What is a computer?

Antikythera mechanism

Differential Analyzer



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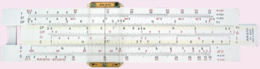

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Laptop

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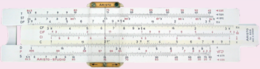

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Differential Analyzer

Laptop

Non-programmable

Programmable

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- If we are not yet able to build those devices (because of technological limitations, etc.), this question can only be answered with mathematical models.
- We have a mathematical model for digital computers – the *Turing machine*.
- But what about for other computing devices?

- In particular, from a theoretical point of view, how do analog computers (differential analyzers) compare in terms of computational power with digital computers (Turing machines)?

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- For example, is it conceivable that one can implement a super analog computer, using some radically new technology not yet developed?

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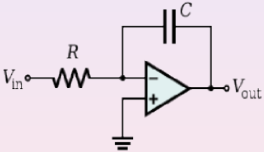

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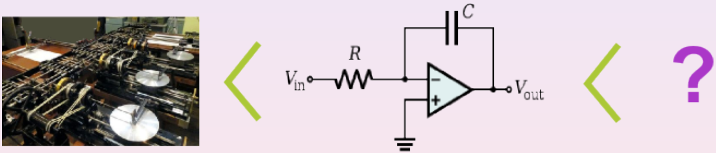

- To answer that question we need a mathematical model for analog computers!

The Shannon machine (1941)

- Shannon worked as an operator on the (mechanical) differential analyzer of Vannevar Bush (better known for heading the U.S. OSRD during World War II, through which almost all wartime military R&D was carried out) to earn some money.

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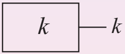
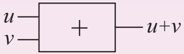
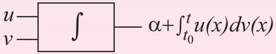
- In 1941 Claude Shannon introduced his *General Purpose Analog Computer (GPAC)* as a mathematical nodel for diferential analyzers.
- It consists of circuits built with four types of basic units:



A constant unit associated to the real value k



An adder unit



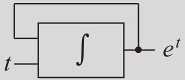
An integrator unit

A multiplier unit

Examples

Example

Compute $y(x) = e^x$ with a GPAC



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

Theorem (Shannon, modern version)

A function is generated by a GPAC if and only if it is the solution of some (vectorial) polynomial initial-value problem

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- Solutions of initial-value problems defined with PIVPs are themselves PIVPs (e.g. the solution of $y_1' = \sin(y_2) + y_1, y_2' = \cos(y_1) + 3$, $y_1(0) = y_2(0) = 1$ is constituted by PIVP functions)

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- Almost all of Classical Physics can be modeled with PIVP functions

In summary

- Shannon introduced a mathematical model (GPAC/PIVPs) for analog computers.
- The PIVP model has nice mathematical properties and can simulate/capture a large class of physical systems.

Main takeaway of this talk

Shannon did for analog computers what Turing did for digital computers.

Some recent results

More recently there has been some research comparing the theoretical computing power of PIVP functions vs. Turing machines.

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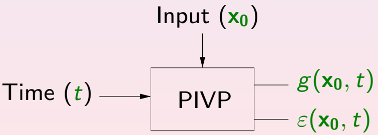
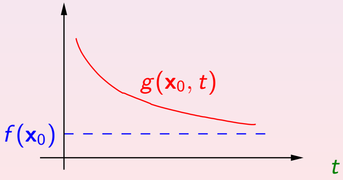
Theorem (Bournez, Campagnolo, Graça, Hainry)

PIVPs and Turing machines are equivalent from a computability perspective.

In the previous result it is assumed that we allow PIVPs to use some time for computation before giving the output.





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As an (unexpected) side effect, the previous result allows us to define the class P of functions computable in polynomial time by using polynomial IVPs and by (polynomially) bounding the length of the solution curve.

Thank you!

Definition (Discrete recognizability)

A language $\mathcal{L} \subseteq \Gamma^*$ is called *analog recognizable* if there exists a vector q of polynomials with two variables and a vector p of polynomials with d variables, both with polynomial-time computable coefficients, and a polynomial $\Omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that for all $w \in \Gamma^*$ there is a (unique) $y : \mathbb{R}_0^+ \to \mathbb{R}^d$ such that for all $t \in \mathbb{R}_0^+$:

- $y(0) = q(\psi(w))$ and $y'(t) = p(y(t))$ ▶ y satisfies a differential equation
- if $|y_1(t)| \geqslant 1$ then $|y_1(u)| \geqslant 1$ for all $u \geqslant t$ ▶ the decision is stable
- if $w \in \mathcal{L}$ (resp. $\notin \mathcal{L}$) and $len_y(0, t) \geqslant \Omega(|w|)$ then $y_1(t) \geqslant 1$ (resp. $\leqslant -1$) ▶ decision
- $len_y(0, t) \geqslant t$ ▶ technical condition

Theorem (An implicit characterization of P - Bournez, Graça, Pouly)

A decision problem (language) $\mathcal{L}$ *belongs to the class P if and only if it is analog recognizable.*